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On the Wavelet Transformation of Fractal Objects

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The wavelet transformation is briefly presented. It is shown how the analysis of the local scaling behavior of fractals can be transformed into the investigation of the scaling behavior of analytic functions over the half-plane near the boundary of its domain of analyticity. As an example, a "Weierstrass-like" fractal function is considered, for which the wavelet transform is related to a Jacobi theta function. Some of the scalings of this theta function are analyzed, and give some information about the scaling behavior of this fractal.

KEY WORDS: Local scaling behavior; oscillatory critical behavior; fractals; Jacobi theta function; behavior on the boundary of analytic function over the half-plane.

1. INTRODUCTION

The purpose of this paper is to present a new method in analyzing fractal objects. These arise in a natural way in physics, e.g., as strange attractors of a dissipative dynamical system (see, e.g., Ref. 2). A typical property of fractals is that they have no natural minimal length scale; they become more and more self-similar as the length scale gets small. We want to give a more precise meaning to this. Throughout this paper we will limit ourselves to fractals represented by bounded continuous functions s over the real line $\mathfrak{R}$, which are in general not differentiable. Let us introduce local variables at any point $(x_0, s(x_0))$ of the graph of s. Consider

$$f_{x_0}(x) = s(x_0 + x) - s(x_0) \tag{1.1}$$

Then self-similarity of s at x_0 will mean the following: if we scale the local variable x with some $a > 0$ ($x \to ax$) and if we rescale at the same time f_{x_0}

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with $a^{-\alpha}$, this process should become stabilized in a nontrivial manner for some $\alpha > 0$ as a gets small:

$$f_{x_0}(ax) \approx a^{\alpha} f_{x_0}(x) \qquad (a \to 0) \tag{1.2}$$

The exponent α will be called the (local) scaling exponent of s at x_0. It measures the relation between the x scale and the s scale at x_0 and is therefore a kind of local dimension of s. In the case where s is the characteristic function of a probability measure μ

$$s(x) = \int \chi_{[0,x]}\, d\mu$$

a theorem of Young[4] shows us that α may be interpreted as a Hausdorff dimension of μ, if α is μ-almost everywhere constant. In this case, when α is essentially independent of x_0, one speaks of a homogeneous fractal, whereas if α depends on x_0 a multifractal description should be used (see, e.g., Ref. 3).

In this paper we will especially be interested in the following family of functions, labeled by a real parameter β:

$$W_\beta(x) = \sum_{n=1,\infty} n^{-\beta} \cos(\pi n^2 x), \qquad \beta > 1 \tag{1.3}$$

Related functions have been studied as examples of fractals.[1,17,19] They are periodic with period 2 and symmetric around $x = 0$. For $\beta < 3$ the series of the formal derivatives of W_β is not absolutely convergent. Figure 1 shows W_β for $\beta = 2$. From this we might guess that W_β is self-similar in the sense presented above; however, a direct verification of (1.2) from (1.3) seems impossible, and so we must transform W_β in an appropriate way. For obvious reasons this transformation should be linear, and involve the notions of scale and position. This is done by the so-called wavelet transformation,[11] which we present later.

This paper is organized as follows: in Section 2 we present the wavelet transformation and give some ideas about how to find the local scaling exponents with the help of this transformation. In Section 3 we give some rigorous results for some special classes of fractals. In Section 4 we show that a suitable wavelet transform of W_β is a Jacobi theta function, and we present some numerical work. In Section 5 we analyze some scalings of W_β with the help of this theta function.

Caution! We will not be able to prove rigorously that W_β satisfies locally (1.2) in whatever sense, but we shall give good reasons to believe

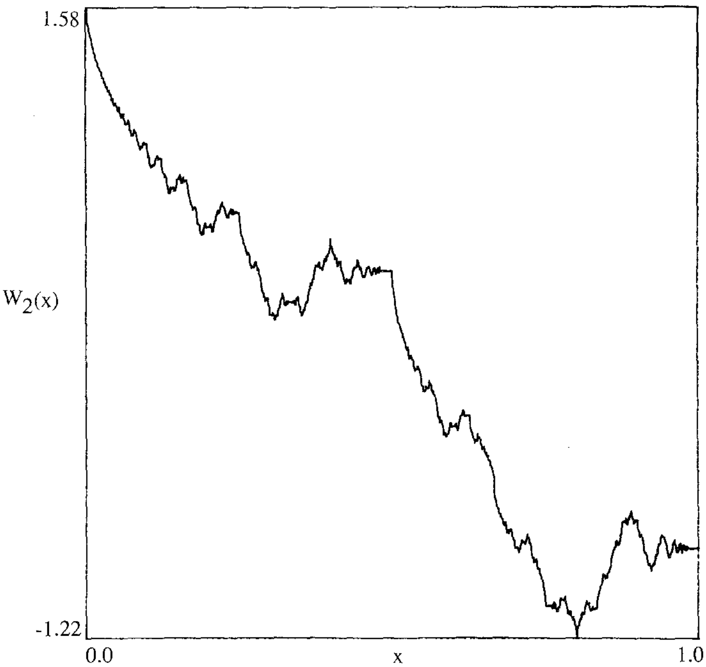


Fig. 1. The function W_β of (1.3) for $\beta = 2$.

this. The reader should keep this in mind if he or she gets the feeling that this paper has two different parts that do not fit together too well. We agree.

2. THE WAVELET TRANSFORMATION

In this section we introduce the wavelet transformation of a real function s over the real line $\mathfrak{R}$. We will limit ourselves to the basic facts that we use in the following. For more detailed information we refer to the rapidly growing literature.[11,13]

We will denote by Fg the Fourier transform of g:

$$Fg(\omega) = \int g(x)\, e^{-i\omega x}\, dx$$

Then we need the following definition:

Definition 1. A function $g \in L^2 \cap L^1$ is called an admissible wavelet[11] if

$$c_g = (2\pi)^{-1} \int |Fg(\omega)|^2\, d\omega/\omega < \infty \tag{2.1}$$

Note that this implies $\int g(x)\,dx = 0$. Then the wavelet transform Ts of a bounded function s with respect to the wavelet g is a function over the half-plane H parametrized by (b, a), $b, a \in \mathfrak{R}$, $a > 0$:

$$\begin{aligned} Ts(b, a) &= \int (1/a)\, \bar{g}([x-b]/a)\, s(x)\, dx \\ &= (2\pi)^{-1} \int \overline{Fg}(a\omega)\, e^{ib\omega}\, Fs(\omega)\, d\omega \end{aligned} \tag{2.2a}$$

It has been shown that this transformation can be inverted for a large class of functions.[11] In this case the inversion formula is given by

$$s(x) = c_g^{-1} \int Ts(b, a)\, g([x-b]/a)\, db\, da/a \tag{2.2b}$$

This transformation is a sort of mathematical microscope, whose magnification is $1/a$, whose position is b, and whose optics is given by the choice of the specific wavelet g.

The following behavior of T under translations and dilations of s is easily verified:

$$\begin{aligned} s(x) \to s(x+c) &\Rightarrow Ts(b, a) \to Ts(b+c, a) \\ s(x) \to s(\lambda x) &\Rightarrow Ts(b, a) \to Ts(\lambda b, \lambda a) \end{aligned} \tag{2.3}$$

From this we might expect that the transform of a function that is self-similar around x_0 with local scaling exponent α, (1.2), will be approximately homogeneous of degree α around $(b, a) = (x_0, 0)$. By an overall translation we can make $x_0 = 0$, and so, using the fact that $\int g = 0$ and (1.2), we can write

$$\begin{aligned} Ts(\lambda b, \lambda a) &= \int (\lambda a)^{-1}\, \bar{g}([x-\lambda b]/[\lambda a])\, s(x)\, dx \\ &= \int (\lambda a)^{-1}\, \bar{g}([x-\lambda b]/[\lambda a])[s(x) - s(0)]\, dx \\ &= \int \bar{g}(x)\, f_0(\lambda[ax+b])\, dx \\ &\approx \lambda^{\alpha} \int \bar{g}(x)\, f_0(ax+b)\, dx \\ &= \lambda^{\alpha}\, Ts(b, a) \end{aligned} \tag{2.4}$$

However, this argument holds only if g decays sufficiently fast at infinity, as can be seen by considering the following example. Let $s(x) = x^{\alpha}$ for $0 < x < 1$; 0 for $x \leqslant 0$; and 1 for $x \geqslant 1$. Then s is approximately homogeneous around $x_0 = 0$. Let the wavelet g behave at infinity as $g \approx x^{-m-1}$. Now let us look at the behavior of the wavelet transform when the position is fixed ($b = 0$) and the scale goes to 0 ($a \to 0$). There will be a local and a global contribution to T:

$$
\begin{aligned}
T(0, a) &= \int_{[0,1]} (1/a)\, \bar{g}(x/a)\, x^{\alpha}\, dx + \int_{[1,\infty]} (1/a)\, \bar{g}(x/a)\, dx \\
&= a^{\alpha} \int_{[0,1/a]} \bar{g}(x)\, x^{\alpha} + \int_{[1/a,\infty]} \bar{g}(x)\, dx \\
&= T_l + T_g
\end{aligned}
$$

The global term behaves always as $T_g \approx a^m$ because of the decay of g at infinity. However, for the local term we obtain $T_l \approx a^{\alpha}$ or $T_l \approx a^m$, depending on whether $\alpha < m$ or $\alpha \geqslant m$, and so the scaling of T is determined by T_l. If $\alpha \geqslant m$, then $T(0, a)$ behaves like a^m and (2.4) does not hold. So we see that the wavelet might be able to read the local scaling exponents $\alpha < m$, but not the others.

3. SOME RIGOROUS RESULTS ABOUT SCALINGS AND WAVELETS

In this section we give some rigorous results about the relation between the local behavior of s—e.g., the scaling exponent α—and the behavior of its wavelet transform for small scales, if s satisfies one of the conditions that we now define.

3.1. Three Classes of Fractals

Definition 2 (ES). A real function s over the real line $\mathfrak{R}$ satisfies at x_0 the exact scaling condition (ES) if there are real constants $\alpha > 0$ and c_+, c_-, not both equal to zero, such that

$$
s(x) = s(x_0) + c_+ |x - x_0|_+^{\alpha} + c_- |x - x_0|_-^{\alpha} + r(x - x_0)
$$

($|x|_+ = 0$ if $x < 0$; $|x|_- = 0$ if $x > 0$; and $|x|$ otherwise) and that the remainder is truly a remainder: $r(x) = o(x^{\alpha})$ $(x \to 0)$.

However, to take into account oscillating critical behavior (see, e.g., Refs. 9 and 10) occurring in the solutions of many functional equations, for instance, equations derived from renormalization group analysis of critical behavior, at least when the renoralization group is discrete, we shall replace the one real exponent α by an infinity of complex exponents $\alpha + in\gamma$, $n \in \mathbb{Z}$. It is essentially the same to replace the constants c_+ and c_- by real functions $v_{+(-)}$ which have the following (discrete) scale invariance: $v_{+(-)}(\beta x) = v_{+(-)}(x)$ for some real $\beta > 0$. In this case β and γ are related via $\beta = \exp(2\pi/\gamma)$. We will call γ the (local) periodic scaling exponent.

Definition 3 (PS). A real function s over the real line satisfies at x_0 the periodic scaling condition (PS) if there are real constants $\alpha > 0$ and $\gamma > 0$ and two real functions v_+ and v_-, which satisfy $v_{+(-)}(\beta x) = v_{+(-)}(x)$ with $\beta = \exp(2\pi/\gamma)$, such that

$$s(x) = s(x_0) + v_+(|x - x_0|)\,|x - x_0|_+^\alpha + v_-(|x - x_0|)\,|x - x_0|_-^\alpha + r(x - x_0)$$

and the remainder r is a remainder: $r(x) = o(x^\alpha)$. In addition, the functions $v_{+(-)}$ should have the following everywhere convergent expansion:

$$v_+(x) = \sum_{n=-\infty,+\infty} d_n^+\,|x|^{i\gamma n}, \qquad v_-(x) = \sum_{n=-\infty,+\infty} d_n^-\,|x|^{i\gamma n}$$

with $\sum_{n=-\infty,+\infty} |d_n^{+(-)}| < \infty$, and d_0^+, d_0^- not both zero.

The minimal regularity of s we shall consider will be the following well-known notion of Hölder continuity:

Definition 4 (H). A real function s over the real line satisfies at x_0 the Hölder condition (H) if there are two real constants $\alpha > 0$ and $c > 0$ such that $|s(x) - s(x_0)| < c\,|x - x_0|^\alpha$ for x close enough to x_0.

In addition, we shall always require that s is bounded ($\|s\|_\infty < \infty$).

We have the following implications: (ES) $\Rightarrow$ (PS) $\Rightarrow$ (H) $\Rightarrow$ the function s is continuous at x_0, since $\alpha > 0$. If s satisfies (PS), then α and γ are uniquely determined by s; if s satisfies (ES), then in addition c_+ and c_- are uniquely determined by s. In the (PS) case the local scaling exponent α can be obtained by the following limit:

$$\alpha = \liminf_{x \to x_0} \log |s(x) - s(x_0)| / \log |x - x_0| \tag{3.1}$$

It is the analog for functions of the exponent previously defined for measures,[3,14] where it is called the local singularity strength. The conditions (ES) and (PS) ensure that s is self-similar around x_0 in the sense of Section 1 (1.2).

3.2. A Class of Wavelets

Special functions require a special treatment, and so we will use as wavelets the following class of functions, which are essentially filters over the positive frequencies.[12] More precisely, we require g to satisfy:

$$\begin{aligned} &\text{(i)} \quad Fg \text{ is real and supp } Fg \subset \mathfrak{R}^+ \\ &\text{(ii)} \quad Fg(\omega) = \omega^m + O(\omega^{m+1}), \quad m > 0 \qquad (\omega \to 0) \\ &\text{(iii)} \quad Fg(\omega) = O(\omega^{-n}) \qquad \forall n > 0 \qquad (\omega \to \infty) \end{aligned} \tag{3.2}$$

These wavelets are all admissible because $m > 0$. The real part of g is even, whereas the imaginary part of g is an odd function, the one being the Hilbert transform of the other. They will give rise to complex-valued wavelet transforms. The behavior of these wavelets at infinity can be found to be $|g(x)| \approx |x|^{-m-1}$. For reasons that shall become clear soon, we introduce the Mellin transforms of all translates of g:

$$M_g(\alpha, \beta) = \int_{[0,\infty]} x^{\alpha-1} g(x-\beta)\, dx, \qquad 0 < \operatorname{Re} \alpha < m+1, \quad \beta \in \mathfrak{R} \tag{3.3}$$

Since $g(-x) = \bar{g}(x)$, and since g has an analytic continuation in the complex upper half-plane H which is continuous on the boundary, one can verify that

$$M_g(\alpha, -\beta) = e^{i\pi\alpha} M_{\bar{g}}(\alpha, \beta) \tag{3.4}$$

3.3. The Transforms of The Special Fractals

As we saw in the previous section, the answer to the question of whether the wavelet transform scales with α when a becomes small might depend on the relation between this local exponent α and the behavior of the wavelet at infinity. We therefore call an exponent α integrable with respect to g if

$$gx^\alpha \in L^1$$

For the wavelets we shall use, this is equivalent to $\alpha < m$.

We now want to look at the wavelet transform for small scale a. The three classes defined above will be treated separately.

3.3.1. Local Hölder Continuity of s. Let s satisfy (H) at x_0 with an exponent α, which is integrable with respect to the wavelet g. Without loss of generality, we may assume that $x_0 = 0$ [see (2.3)]. Then, using the fact that $\int g\, dx = 0$, we can write

$$Ts(b, a) = \int (1/a)\, \bar{g}([x-b]/a)[s(x) - s(0)]\, dx$$

Changing x in ax, we can estimate the modulus of Ts:

$$|Ts(b, a)| \leqslant \int |g(x-b/a)|\, |s(ax)-s(0)|\, dx$$

We now want to separate the local and the global contributions of Ts. Because s is Hölder continuous at $x_0=0$, there is a $\delta>0$ such that $|x|<\delta$ implies $|s(x)-s(0)|<c|x|^\alpha$ with some real $c>0$. We will use this δ to cut the integral into two parts:

$$|Ts(b, a)| \leqslant \left(\int_{|ax|<\delta} + \int_{|ax|\geqslant\delta}\right) |g(x-b/a)|\, |s(ax)-s(0)|\, dx = T_l + T_g$$

In the local part T_l we majorize the integrand with the help of the Hölder condition, whereas in the global part T_g the boundedness of s will be used. Since $gx^\alpha \in L^1$, we obtain

$$T_l \leqslant ca^\alpha \int_{|ax|<\delta} |g(x-b/a)|\, |x|^\alpha\, dx$$

$$\leqslant ca^\alpha \int |g(x-b/a)|\, |x|^\alpha\, dx$$

$$T_g \leqslant 2\,\|s\|_\infty \int_{|x|<\delta/a} |g(x-b/a)|\, dx$$

We denote by $H_\delta(x)$ the cone in the half-plane with opening angle $\pi-2\delta$: $H_\delta(x)=\{(b, a)\in H \mid \delta<\arg(b+ia)<\pi-\delta\}$. Then, for all $\delta>0$, small enough, we have $T_l=O(a^\alpha)$ and $T_g=O(a^m)$ $(a\to 0)$ uniformly in $H_\delta(0)$. So, using the translation invariance (2.3) of Ts, we find that the local Hölder continuity of degree $\alpha<m$ of s at x_0 implies the same kind of regularity of Ts, provided the point $(x_0, 0)$ situated at the boundary of the open half-plane is approached in a nontangential way:

$$\forall\delta>0 \Rightarrow Ts(x_0+b, a)=O(a^\alpha) \qquad \text{for} \quad a\to 0 \quad \text{and} \quad (b, a)\in H_\delta(0) \tag{3.5}$$

Let now the local exponent be nonintegrable with respect to the wavelet g: $\alpha \geqslant m$. Then the global term is still $T_g=O(a^m)$, but the local term is found to be $T_l=O(a^m)$ or $T_l=O(a^m \log a)$, depending on whether $\alpha>m$ or $\alpha=m$. The point $(x_0, 0)$ should again be approached in a nontangential manner. All this together shows that not all of the local regularity of s at x_0 can be found in Ts if the wavelet does not decay sufficiently fast at infinity. We have instead

$$\begin{aligned}(b, a)\in H_\delta(0) \Rightarrow Ts(x_0+b, a) &= O(a^m \log a) \quad (a\to 0) \quad \text{if} \quad \alpha=m \\ &= O(a^m) \qquad\quad (a\to 0) \quad \text{if} \quad \alpha>m\end{aligned} \tag{3.6}$$

So we see that the wavelets work as a sort of filter: only the integrable exponents will be distinguished by the wavelet, whereas the nonintegrable exponents might give rise to the same scaling of Ts, due to the decay of g at infinity.

3.3.2. Exact Scaling Condition. Let s satisfy (ES) at x_0 and let the local exponent α be integrable with respect to the wavelet g. Again we may assume that $x_0 = 0$. Starting from

$$Ts(b, a) = \int (1/a)\, \bar{g}([x-b]/a)[s(x) - s(0)]\, dx$$

we change x in ax. Using the property (ES) of s, we obtain

$$\begin{aligned} Ts(b, a) &= \int \bar{g}(x - b/a)[s(ax) - s(0)]\, dx \\ &= \int \bar{g}(x - b/a)[c_+ |ax|_+^\alpha + c_- |ax|_-^\alpha + r(ax)]\, dx \\ &= C(b/a)\, a^\alpha + R \end{aligned} \tag{3.7a}$$

with

$$\begin{aligned} C(u) &= c_+ M_{\bar{g}}(\alpha+1, u) + c_- M_g(\alpha+1, -u) \\ &= (c_+ - e^{i\pi\alpha} c_-)\, \overline{M_g(\alpha+1, u)} \end{aligned} \tag{3.7b}$$

since α is real, (3.4).

We now want to estimate the remainder R. First note that r can be written as $r(x) = |x|^\alpha \rho(x)$ with ρ satisfying:

(i) $\forall \varepsilon > 0,\ \exists \delta > 0$ such that $|x| < \delta \Rightarrow |\rho(x)| < \varepsilon$,

(ii) $\|\rho\|_\infty < \infty$.

The first is clear because $r = o(x^\alpha)$, and the second follows from $\|s\|_\infty < \infty$. We want to show that $|R|/a^\alpha$ can be made arbitrary small. So let $\varepsilon > 0$ be given. A first estimate is immediate:

$$|R|/a^\alpha \leqslant \int |g(x - b/a)|\, |x|^\alpha\, |\rho(x)|\, dx$$

As before, we will use δ to separate the local and the global contributions in the remainder, and so

$$|R|/a^\alpha \leqslant \left(\int_{|ax| < \delta} + \int_{|ax| \geqslant \delta} \right) |g(x - b/a)|\, |x|^\alpha\, |\rho(x)|\, dx$$

In the first integral we majorize $|\rho|$ by ε, and in the second by $\|\rho\|_\infty$. Since $gx^\alpha \in L^1$, we obtain

$$|R|/a^\alpha \leqslant \varepsilon \int_{|ax|<\delta} |g(x-b/a)|\,|x|^\alpha\,dx + \|\rho\|_\infty \int_{|ax|\geqslant\delta} |g(x-b/a)|\,|x|^\alpha$$
$$\leqslant c_{te}\varepsilon + o(a)$$

But ε was arbitrary and therefore $R = o(a^\alpha)$, whenever $(x_0, 0)$ is approached in a nontangential way as before $[(b, a) \in H_\delta(0)]$.

All this together shows that the wavelet transform of a bounded function s satisfying (ES) at x_0 with an exponent α is approximately homogeneous at $(x_0, 0)$, provided the exponent α is integrable with respect to the wavelet g:

$$Ts(x_0 + \lambda b, \lambda a) = \lambda^\alpha Ts(x_0 + b, a) + o(\lambda^\alpha), \qquad \lambda \to 0 \tag{3.8}$$

We still should make sure that the function C in (3.7) does not vanish for all u. Since $c_{+(-)}$ are real constants, a sufficient condition will be that the exponent α is not an integer.

It was found[12] that it is useful to consider the modulus and the phase of the wavelet transform separately. Let $\Phi = \arg Ts$; then we see that Φ is to leading order a function of the direction b/a only: $\Phi = \Phi_0(b/a) + \cdots$, and so the lines of constant phase will pass through the point $(x_0, 0)$ when the singularity of s is situated at x_0. If the point $(x_0, 0)$ is approached from above $(b/a = 0)$, then we see with the help of (3.7b) that the leading term Φ_0 contains some information about the local symmetry of s at x_0. So $\Phi_0(x_0, 0) = 0 \bmod \pi$ implies that s is approximately even around x_0, whereas $\Phi_0(x_0, 0) = \pi/2 \bmod \pi$ indicates that s is odd around x_0.

3.3.3. The Periodic Scaling Condition. Let s satisfy (PS) at x_0 and let the local exponent be integrable with respect to the wavelet g. From the definition of (PS) it is clear that locally s can be seen as a sum of functions satisfying the analog of condition (ES), but where the real exponent α is replaced by the complex exponent $\alpha + in\gamma$, $n \in \mathbb{Z}$. Since this sum is by definition absolutely convergent, we may treat it term by term. So the same considerations as before lead us to

$$Ts(x_0 + b, a) = v(b/a, a)\, a^\alpha + R(b, a) \tag{3.9a}$$

with a complex-valued function

$$v(t, u) = \sum_{n=-\infty,+\infty} d_n(t)\, u^{in\gamma}, \qquad t, u \in \mathfrak{R}, \quad u > 0 \tag{3.9b}$$

and the functions d_n are given by

$$d_n(u) = d_n^+ M_{\bar{g}}(\alpha + 1 + in\gamma, u) + d_n^- M_g(\alpha + 1 + in\gamma, -u) \tag{3.9c}$$

The remainder is again a remainder when the point $(x_0, 0)$ is approached in a nontangential way: $(b, a) \in H_\delta(0) \Rightarrow R = o(a^\alpha)$ for any $\delta > 0$, small enough.

So the wavelet transform is again approximately homogeneous of degree α, but now in an oscillatory manner, due to the oscillatory behavior of s at x_0:

$\forall (b, a) \in H_\delta(0)\ \exists$ function f such that:

$$Ts(x_0 + \lambda b, \lambda a) = f(\lambda)\, \lambda^\alpha Ts(x_0 + b, a) + o(\lambda^\alpha) \tag{3.10}$$

All these complex-valued functions f will have the same (discrete) scale invariance as the local functions $v_{+(-)}$ associated to s by the condition (PS): $f(\beta\lambda) = f(\lambda)$ for $\beta = \exp(2\pi/\gamma)$.

Consider again the phase $\Phi = \arg Ts$. We suppose that Ts is different from zero along the straight line passing through $(x_0, 0)$, so that Φ can be made a continuous function along this line. Then we see that Φ will turn, when we approach the point $(x_0, 0)$:

$$\Phi(x_0 + \beta b, \beta a) = \Phi(x_0 + b, a) + 2\pi n \tag{3.11}$$

with some $n \in \mathbb{Z}$.

3.3.4. The Results. We have seen that the local properties of s at x_0 can be found as local properties of Ts at $(x_0, 0)$ when $(x_0, 0)$ is approached in a nontangential way. So we will define the local scaling exponent α_T of Ts at $(x_0, 0)$ for any bounded function s as

$$\alpha_T = \inf_{\delta > 0} \ \liminf_{(b,a) \to (x_0, 0),\, (b,a) \in H_\delta(x_0)} \ \log |T(b, a)| / \log a \tag{3.12}$$

If s satisfies (PS), then the inf's will actually be reached. To take into account oscillatory critical behavior of Ts we define the local periodic scaling exponent γ_T as follows:

$$\gamma_T = \inf_{\delta > 0} \ \liminf_{(b,a) \to (x_0, 0),\, (b,a) \in H_\delta(x_0),\, Ts(b,a) \neq 0} \ [\Phi(b, a) / \log a] \tag{3.13}$$

Here the phase $\Phi = \arg Ts$ is supposed to be a continuous function along the way (b, a). This can always be achieved, since Ts is different from zero along this way.

Then we can summarize our results in the following theorems:

Let s be a bounded, real function over the real line. Let g be a wavelet satisfying (3.2) with $m > 0$, and let Ts be the wavelet transform of s with respect to g_m. We then have the following results for the three classes of fractals (H), (ES), (PS):

Theorem 1 (H case). Let s be Hölder continuous (H) at x_0 with an exponent $\alpha > 0$. Let α_T be the local scaling exponent of Ts at x_0. Then:

(i) $\alpha < m \Rightarrow \forall \delta > 0$: $Ts(x_0 + b, a) = O(a^\alpha)$ for $a \to 0$ and $(b, a) \in H_\delta(0)$.

(ii) $\alpha < m \Rightarrow \alpha_T \geqslant \alpha$; $\alpha \geqslant m \Rightarrow \alpha_T \geqslant m$.

Theorem 2 (ES case). Let s satisfy (ES) at x_0 with a noninteger local scaling exponent $\alpha < m$. Let α_T be the local scaling exponent of Ts at x_0, and γ_T its periodic scaling exponent. Then:

(i) $\forall (b, a) \in H$: $Ts(x_0 + \lambda b, \lambda a) = \lambda^\alpha Ts(x_0 + b, a) + o(\lambda^\alpha)$ $(\lambda \to 0)$.

(ii) $\alpha_T = \alpha$, $\gamma_T = 0$.

Theorem 3 (PS case). Let s satisfy (PS) at x_0 with a noninteger local scaling exponent $\alpha < m$, and periodic scaling exponent γ. Let α_T be the local scaling exponent of Ts at x_0, and γ_T its periodic scaling exponent. Then:

(i) $\forall (b, a) \in H$, $\exists$ function f such that
$Ts(x_0 + \lambda b, \lambda a) = f(\lambda)\, \lambda^\alpha Ts(x_0 + b, a) + o(\lambda^\alpha)$
and $f(\beta\lambda) = f(\lambda)$ for $\beta = \exp(2\pi/\gamma)$.

(ii) $\alpha_T = \alpha$, $\gamma_T = n\gamma$, for some $n \in \mathbb{Z}$.

Remark. We were not able to prove rigorously that a specific scaling behavior of the transform Ts implies a scaling behavior of s of the same kind. But nevertheless we strongly believe that there is such an intimate relation. The rest of this paper should be understood in that sense.

4. THE WAVELET TRANSFORM OF A SPECIAL FAMILY OF FRACTALS

We now give an explicit example of wavelet transforms of some fractal functions. As mentioned in the introduction, we will be interested in the following family of functions:

$$W_\beta(x) = \sum_{n=1,\infty} n^{-\beta} \cos(\pi n^2 x), \qquad \beta > 1 \tag{4.1}$$

Related functions have been studied as examples of fractals.[1,17,19] They are periodic with period 2 and symmetric around $x = 0$. For $\beta < 3$ the series of the formal derivatives of W_β seems to diverge. We do not know whether the functions $W_\beta(x)$ satisfy one of the conditions we have treated in the previous section, but Fig. 1 may convince the reader that at least they might be self-similar. Therefore, it will be interesting to analyze these functions with the help of an appropriate wavelet transformation.

4.1. The Transformation Wavelets

Following Ref. 15, we will use as wavelets the following functions, given in the Fourier space by

$$Fg_m(\omega) = |\omega|_+^m \, e^{-\omega} \tag{4.2}$$

These wavelets are filters over the positive frequencies of the kind we have used in the previous section. A simple calculation yields

$$g_m(x) = (2\pi)^{-1} \, \Gamma(m+1)(1-ix)^{-m-1} \tag{4.3}$$

and the Mellin transforms (3.3) are

$$M_{g_m}(\alpha, \beta) = (2\pi)^{-1} \, \Gamma(\alpha) \, \Gamma(m+1-\alpha) \, e^{i\pi\alpha/2}(1+i\beta)^{\alpha-m-1} \tag{4.4}$$

Let s be the function that we want to transform. Then this family of wavelets will give rise to wavelet transforms Ts, which are of the following special form[15]:

$$\begin{aligned} Ts(b, a) &= (2\pi)^{-1} \int \overline{Fg}(a\omega) \, e^{ib\omega} Fs(\omega) \, d\omega \\ &= (2\pi)^{-1} \int_{[0,\infty]} (a\omega)^m \, e^{-a\omega + ib\omega} Fs(\omega) \, d\omega \\ &= (2\pi)^{-1} \, a^m \int_{[0,\infty]} \omega^m e^{i(b+ia)\omega} Fs(\omega) \, d\omega \end{aligned} \tag{4.5}$$

We now write $b + ia = z$ and so the half-plane on which Ts is defined is the complex upper half-plane. The transform itself can be written as

$$Ts(z) = (imz)^{-m} \, D(z) \tag{4.6}$$

where D is an analytic function of the complex upper half-plane. This reflects the fact that the wavelet g_m is essentially the mth derivative of a Poisson kernel (e.g., Ref. 16). The scale parameter is now the imaginary part of z, and so the study of the small-scale behavior of s is transformed into the study of the behavior of an analytic function near the boundary of the upper half-plane, which for a fractal function will in general coincide with the boundary of its analyticity domain.

4.2. The Wavelet Transformation of W_β

Now we come to calculate the transform $T_{\beta,m}$ of W_β with respect to these wavelets. The Fourier transform of W_β is a sum of δ-functions, and so we obtain

$$T_{\beta,m}(z) = \frac{1}{2} \pi^m (imz)^m \sum_{n=1,\infty} n^{2m-\beta} \, e^{i\pi n^2 z} \tag{4.7}$$

Things will become much nicer if we choose $\beta = m/2$. Then the transform $T_\beta = T_{\beta,\beta/2}$ is essentially a Jacobi theta function:

$$T_\beta(z) = \frac{1}{4}\pi^{\beta/2}(imz)^{\beta/2}\,[\vartheta(z) - 1] \tag{4.8a}$$

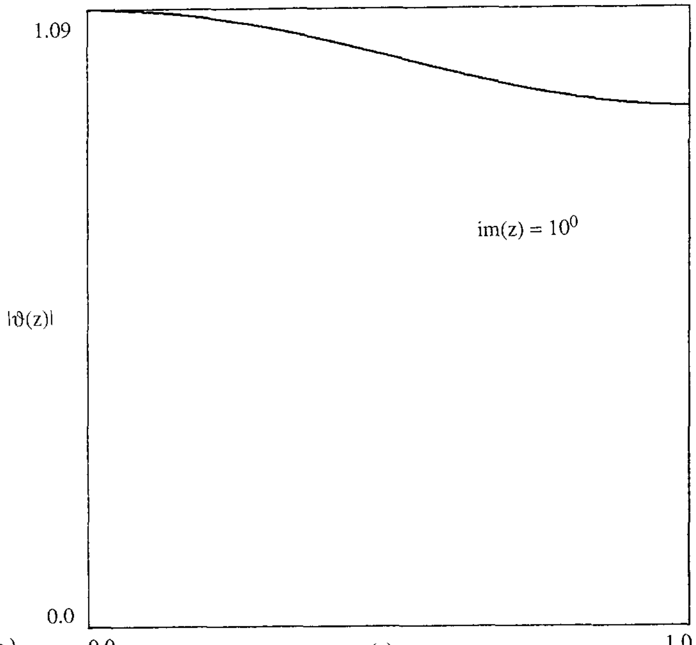
with

$$\vartheta(z) = \sum_{n=-\infty,+\infty} e^{i\pi n^2 z} \tag{4.8b}$$

In order to analyze the local scaling behavior of W_β, we should explore the scaling behavior of the Jacobi theta function (4.8b) near the real line, which is the boundary of its analyticity domain. This will be done in the next section. First we shall give some numerical results.

4.3. Some Numerical Results

Figure 2 shows $|\vartheta|$ versus the position parameter Re(z), the scale parameter Im(z) being fixed at different values ranging from 1.0 to 10^{-6}.



(a)

Fig. 2. The absolute value of the theta function $|\vartheta(z)|$ of (4.8b) versus Re(z). The imaginary part of z is fixed at different values ranging from 1 to 10^{-6}. Since ϑ is essentially a wavelet transform of W_β (Fig. 1), Im(z) can be interpreted as a scale parameter.

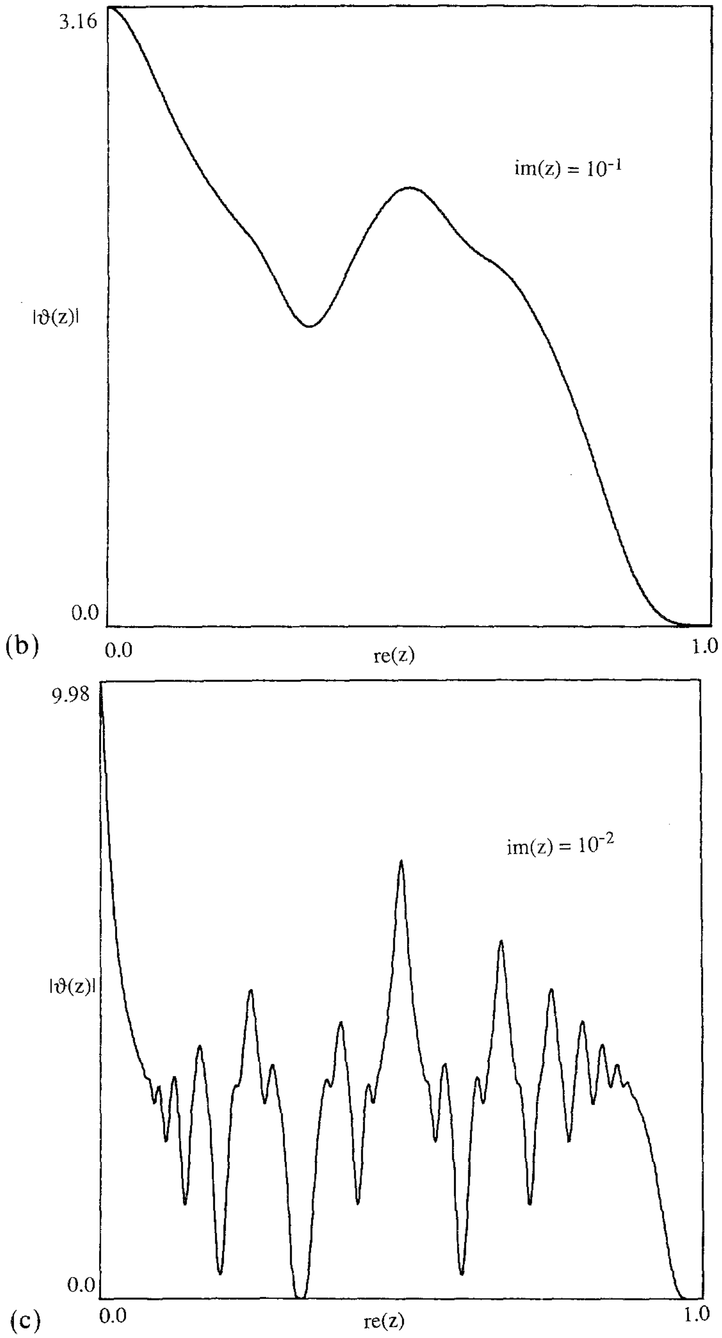


Fig. 2 (*continued*)

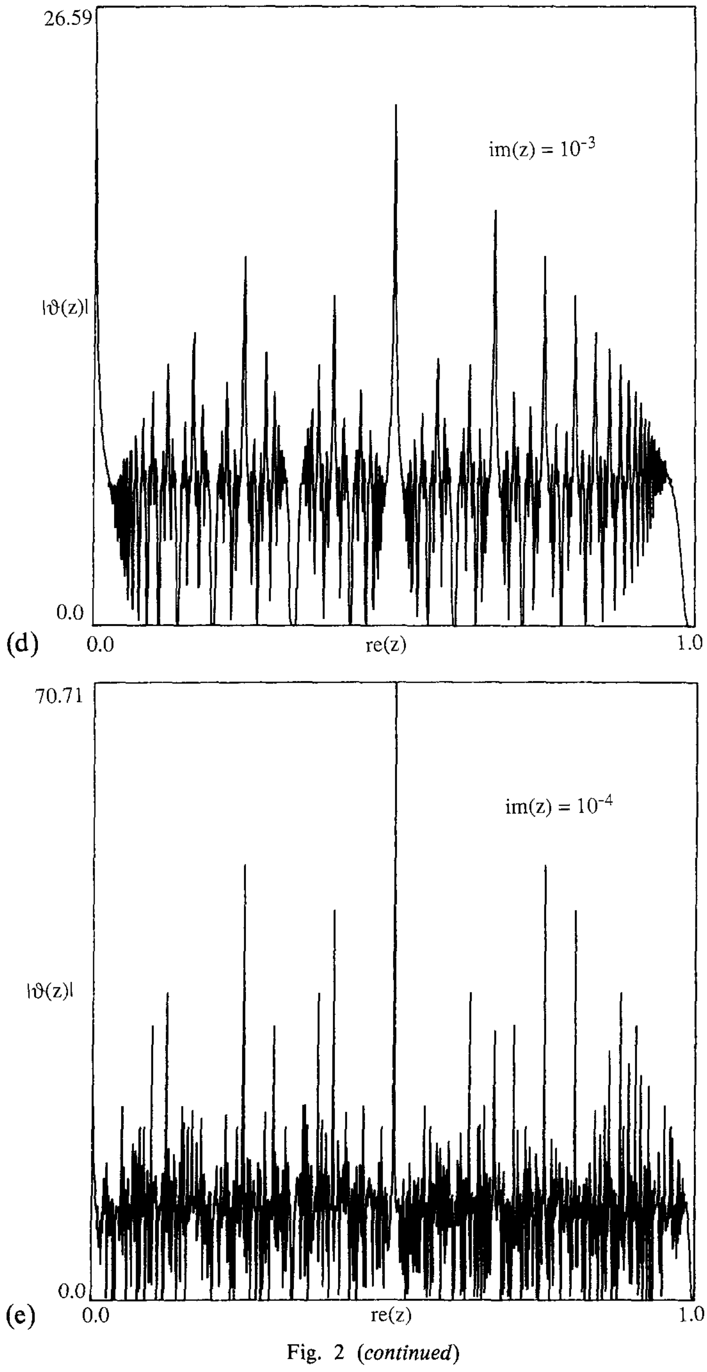
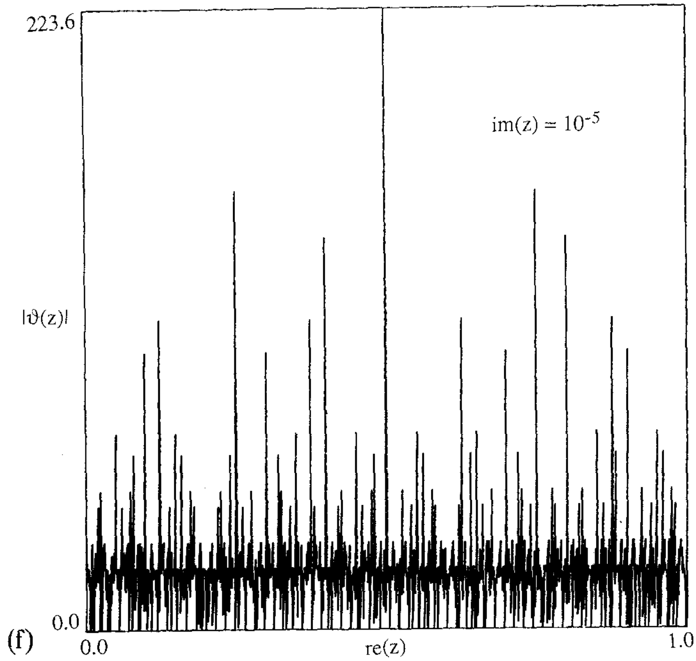


Fig. 2 (*continued*)



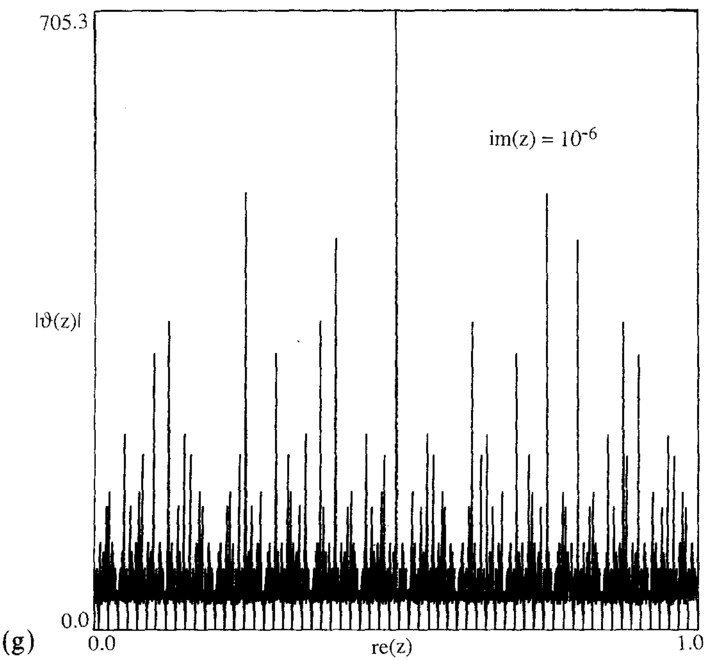


Fig. 2 (*continued*)

As the scale becomes small, the pictures seem to become stabilized when $|\vartheta|$ is rescaled by a factor depending only on the scale $\mathrm{Im}(z)$ in such a way that the peak at $\mathrm{Re}(z) = 1/2$ fits into the picture. Notice the change of the length scale at the $|\vartheta|$ axis. More exactly, let us look at the following family h_λ of real functions over the real line:

$$h_\lambda(x) = \lambda^{-\alpha} |\vartheta(x + i\lambda)| \tag{4.9}$$

where the scaling exponent α is fixed for each λ by requiring that $h_\lambda(1/2) = 1$. Then, what actually is observed numerically is that the limit $h_\lambda(\lambda \to 0)$ exists and has the shape of Fig. 2g. To put it differently, the renormalization procedure (4.9) has a nontrivial fixed point. This gives numerical evidence that ϑ—and hence the wavelet transform of W_β—is locally of the form (3.8); that is, locally homogeneous. So the peaks we see for small scale might be situated at the real points for which the local scaling exponent of s is identical with the one at $x_0 = 1/2$. Since there are no points at which h_λ diverges as λ goes to zero, this exponent might correspond to the smallest scaling exponent that can be found in W_β. The

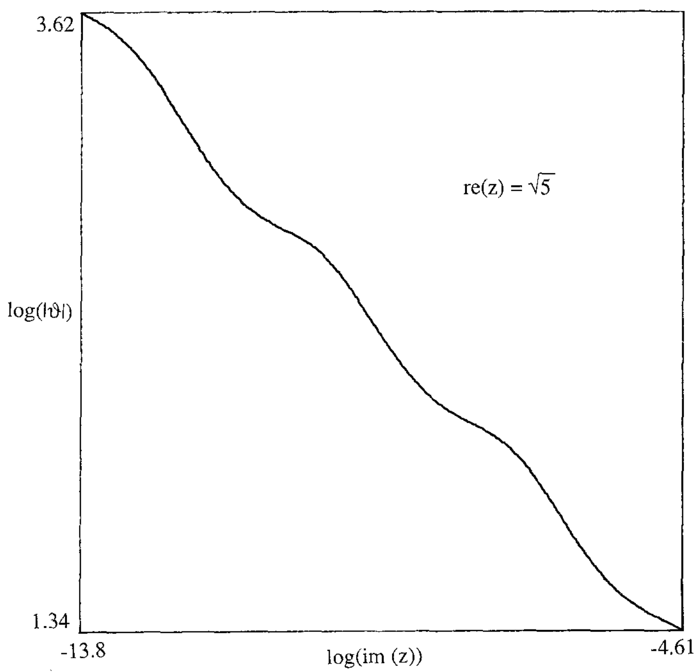


Fig. 3. Plot of $\log|\vartheta(z)|$ versus $\log(\mathrm{Im}\, z)$, where $\mathrm{Re}\, z$ is fixed at $\sqrt{5}$. This shows that the Jacobi theta function (4.8b) shows oscillatory critical behavior when the boundary of its analyticity domain is approached in this nontangential way.

height of the peaks might then be related to the local constants c_+ and c_- at these points via (3.7).

Figure 3 shows $\log |\vartheta(x+i\lambda)|$ versus $\log \lambda$, where the position is fixed at $x=\sqrt{5}$. The oscillations around a straight line with slope $\approx -1/4$ might indicate that at this point W_β shows oscillatory critical behavior.

In the following section we will explain theoretically these numerical results.

5. SOME SCALINGS OF THE THETA FUNCTION

In this section we study the behavior of ϑ in the neighborhood of the real axis. Specifically, we show that at one class of rational points ϑ is governed by an exponent $-1/2$, at another by an exponent ∞, whereas there is a well-defined set of irrationals where ϑ is governed by an imaginary exponent, whose real part is $-1/4$.

We define for ϑ the analog of the scaling exponents of the wavelet transforms (3.12), (3.13). (We denote by H the complex upper half-plane.) Let $H_\delta(x)$ be the cone $H_\delta(x)=\{z\in H \mid \delta<\arg(z-x)<\pi-\delta\}$. Then

$$\alpha_\vartheta(x)=\inf_{\delta} \liminf_{z\to x, z\in H_\delta(x)} \log \vartheta(z)/\log(z-x) \tag{5.1}$$

The limit should hold for the real and the imaginary parts separately. For the sets of x values that we shall consider, the limit processes can be replaced by the simple limit. However, (5.1) ensures that the exponent is well-defined for arbitrary x.

Since ϑ is never zero in H,[7] and H is simply connected, $\log \vartheta$ can be made a holomorphic function in H. A moment's reflection shows that the scaling exponent α_T of (3.12) of the wavelet transform T_β of W_β is related to α_ϑ via $\alpha_T=\beta/2+\operatorname{Re}\alpha_\vartheta$, whereas the periodic scaling exponent γ_T of (3.13) satisfies $\gamma_T=\operatorname{Im}\alpha_\vartheta$.

Without any supplementary knowledge about ϑ, we can give a lower bound for the real part of the scaling exponent α_ϑ of ϑ: for any $\beta>3$, the series (4.1) represents an everywhere differentiable function, and so W_β satisfies (H) at every point with a local scaling exponent $\alpha\geqslant 1$. On the other hand, from the theorem we proved in Section 3 it follows that $\alpha_T\geqslant 1$. Using the relation between α_T and α_ϑ we find

$$\operatorname{Re}\alpha_\vartheta\geqslant -1/2 \tag{5.2}$$

It actually is known that $\vartheta(z)=O(|\operatorname{Im} z|^{-1/2})$.[6]

5.1. Some Basic Facts about ϑ

To get more detailed information, we must work a little harder. Denote by G the modular group

$$G = \{z \to (az+b)/(cz+d) \mid a, b, c, d \text{ are integers, } ad-bc=1\}$$

Every element of G is a meromorphic function, which leaves invariant the upper and the lower half-planes, the real axis, and the rationals. We denote by D the differentiation operator. The derivative Dg of any $g \in G$ is easily calculated:

$$g \in G, \qquad g(z) = (az+b)/(cz+d) \Rightarrow Dg(z) = 1/(cz+d)^2 \tag{5.3}$$

The group G is (not freely) generated by two elements, the translation T and the negative inversion U,[5]

$$G = \langle T, U \rangle, \qquad T: z \to z+1, \qquad U: z \to -1/z \tag{5.4}$$

The following transformation formulas for ϑ are known[7]:

$$\vartheta(T^2 z) = \vartheta(z) \tag{5.5a}$$

$$\vartheta(Uz) = (-iz)^{1/2}\, \vartheta(z) \tag{5.5b}$$

The square root is uniquely determined by the fact that $U(i) = i$.

The subgroup of G generated by T^2 and U is called[18] the theta group G_ϑ. It is a nonnormal subgroup of G of index 3 $(G: G_\vartheta = 3)$ and so there are three cosets of G_ϑ in G. From the fundamental region of G_ϑ,[18] one can find three coset representatives, and so

$$G = G_\vartheta \cup G_\vartheta T^{-1} \cup G_\vartheta U T^{-1} \tag{5.6}$$

Since any element g of G_ϑ can be written as a finite product of U and T^2, we can apply successively (5.5), and obtain the following covariance of ϑ under the action of $g \in G_\vartheta$:

$$\vartheta(g(z)) = f_g(z)\, \vartheta(z) \tag{5.7}$$

where the multiplier $f_g(z)$ is uniquely defined by the following:

$$\text{(i)} \quad g_1, g_2 \in G_\vartheta \to f_{g_1 g_2}(z) = f_{g_1}(g_2(z))\, f_{g_2}(z) \tag{5.8a}$$

$$\text{(ii)} \quad f_{T^2}(z) = 1, \quad f_U(z) = (-iz)^{1/2} \tag{5.8b}$$

Equation (5.8a) is a cocycle condition. It is similar to the chain rule for the derivations: $Dg_1 \circ g_2(z) = Dg_1(g_2(z))\, Dg_2(z)$. Therefore, since $DT^2(z)^{-1/4} =$

$|f_{T^2}(z)|^{-4}$ and $f_U(z)=\zeta\,|DU(z)|^{-1/4}$, with $\zeta^8=1$, we have the following relation between Dg and f_g:

$$f_g(z)=\zeta\,|Dg(z)|^{-1/4} \qquad \text{with some} \quad \zeta^8=1 \tag{5.9}$$

For $\log\vartheta$, the multiplicative covariance (5.8) will be transformed in the following additive transformation formulas for $g\in G_\vartheta$:

$$\log\vartheta(g(z))=\log\vartheta(z)+r_g(z) \tag{5.10}$$

where r_g is uniquely defined by:

$$\text{(i)} \quad g_1,\, g_2\in G_\vartheta \to r_{g_1g_2}(z)=r_{g_1}(g_2(z))+r_{g_2}(z) \tag{5.11a}$$

$$\text{(ii)} \quad r_{T^2}(z)=0, \quad r_U(z)=\tfrac{1}{2}\log(z)-i\pi/4 \tag{5.11b}$$

The first is clear since ϑ has no zeros in the half-plane H, and since $\vartheta\approx 1$ for $\operatorname{Im} z\to\infty$, as easily seen from (4.8b). The second follows from (5.8b) and the fact that $U(i)=i$, which also determines the branch of the logarithm.

Now we have tools to study the theta function near the real axis.

5.2. Some Transformation Formulas

In a first lemma we show that the scaling exponent $\alpha_\vartheta(x)$ of ϑ at x, (5.1), is invariant under the action of G_ϑ.

Lemma 1. Let $g\in G_\vartheta$ and $x\in\mathfrak{R}$ such that $g(x)\neq\infty$. Then $\alpha_\vartheta(g(x))=\alpha_\vartheta(x)$.

Proof. It is enough to consider the behavior of α_ϑ under the generators of G_ϑ. Because of the periodicity of ϑ, it is clear that $\alpha_\vartheta(T^2(x))=\alpha_\vartheta(x)$. So we consider U for $x\neq 0$. For any $\delta>0$ there is a $\delta'>0$ such that $z\in H_\delta(x)\Rightarrow U(z)\in H_{\delta'}(Ux)$ for z close enough to x. Because $U^{-1}=U$, this statement holds also with $U(z)$ and z exchanged. So we can write

$$\alpha_\vartheta(Ux)=\inf_{\delta}\ \liminf_{z\to x,\, z\in H_\delta(x)}\ \log\vartheta(Uz)/\log(Uz-Ux)$$

In the following we will abbreviate the above limit by LIM. Using the covariance (5.10) of $\log\vartheta$, we obtain

$$\alpha_\vartheta(Ux)=\mathrm{LIM}[r_U(z)+\log\vartheta(z)]/\log(Uz-Ux)$$

Since $r_U(z)$ of (5.11b) is bounded in a neighborhood of $x \neq 0$, it does not contribute. Multiplying by $1 = \log(z-x)/\log(z-x)$ yields

$$\alpha_\vartheta(Ux) = \mathrm{LIM}[\log \vartheta(z) \log(z-x)]/[\log(Uz-Ux)\log(z-x)]$$

Now U has a nonvanishing first derivative at x, and therefore $\log(Uz-Ux)/\log(z-x) \to 1$, and so

$$\alpha_\vartheta(Ux) = \mathrm{LIM} \log \vartheta(z)/\log(z-x) = \alpha_\vartheta(x) \quad \blacksquare$$

From the intuitive idea that ϑ is locally of the form (3.8), that is, locally homogeneous, it might be interesting to consider the following (complex) quantity:

$$C_\vartheta(x) = \inf_{\delta} \liminf_{z \to x, z \in H_\delta(x)} [\vartheta(z)/(z-x)^{\alpha_\vartheta(x)}] \tag{5.12}$$

Here all the limits should hold for the real and the imaginary parts separately. The next lemma will show us how this local constant changes under the action of the theta group G_ϑ.

Lemma 2. Let $g \in G_\vartheta$ and $x \in \Re$ such that $g(x) \neq \infty$. Then $C_\vartheta(g(x)) = p_g(x)\, C_\vartheta(x)$, and the multiplier p_g is uniquely defined by:

$$\text{(i)} \quad g_1, g_2 \in G_\vartheta \to p_{g_1 g_2}(x) = p_{g_1}(g_2(x))\, p_{g_2}(z) \tag{5.13a}$$

$$\text{(ii)} \quad p_{T^2}(x) = 1, \quad p_U(x) = e^{-i\pi\, \mathrm{sign}(x)/4} |x|^{1/2 + 2\alpha_\vartheta(x)} \tag{5.13b}$$

Proof. Again it is enough to consider the behavior of C_ϑ under the generators of G_ϑ. Since ϑ is periodic, (5.5), it is clear that $C_\vartheta(T^2 x) = C_\vartheta(x)$. Therefore we only need to prove that

$$C_\vartheta(Ux) = e^{-i\pi\, \mathrm{sign}(x)/4} |x|^{1/2 + 2\alpha_\vartheta(x)} C_\vartheta(x) \qquad \text{for} \quad x \neq 0$$

As before, for any $\delta > 0$ there is a δ' such that $z \in H_\delta(x) \Rightarrow U(z) \in H_{\delta'}(Ux)$ for z close enough to x and vice versa. So we can write

$$C_\vartheta(Ux) = \inf_{\delta} \liminf_{z \to x, z \in H_\delta(x)} [\vartheta(Uz)/(Uz-Ux)^{\alpha_\vartheta(Ux)}]$$

We will again use the abbreviation LIM for this limit. Since α_ϑ is invariant, and ϑ is covariant under U, we obtain

$$C_\vartheta(Ux) = \mathrm{LIM}[f_U(z)\, \vartheta(z)]/(Uz-Ux)^{\alpha_\vartheta(x)}$$

Now multiplying with $1=(z-x)^{\alpha_\vartheta(x)}/(z-x)^{\alpha_\vartheta(x)}$, and using the fact that U is differentiable and f_U is continuous at x, we obtain

$$\begin{aligned} C_\vartheta(Ux) &= \mathrm{LIM}[f_U(z)\,\vartheta(z)(z-x)^{\alpha_\vartheta(x)}]/[(Uz-Ux)^{\alpha_\vartheta(x)}(z-x)^{\alpha_\vartheta(x)}] \\ &= \lim_{z\to x}[f_U(z)\,DU(z)^{-\alpha_\vartheta(x)}]\,\mathrm{LIM}[\vartheta(z)/(z-x)^{\alpha_\vartheta(x)}] \\ &= C_\vartheta(x)\lim_{z\to x} f_U(z)\,DU(z)^{-\alpha_\vartheta(x)} \\ &= e^{-i\pi\,\mathrm{sign}(x)/4}\,|x|^{1/2+2\alpha_\vartheta(x)}\,C_\vartheta(x) \quad \blacksquare \end{aligned}$$

If ϑ is locally approximately homogeneous, it will be interesting to consider the behavior of the remainder under G_ϑ. This will be done in the next lemma.

Lemma 3. Let ϑ be approximately homogeneous at $x\in\mathfrak{R}$:

$$\forall\delta>0, \quad z\in H_\delta(0)\Rightarrow \vartheta(x+z)=C_\vartheta z^{\alpha_\vartheta}+O(z^{\alpha_\vartheta+1})$$

Let $g\in G_\vartheta$, and $g(x)\neq\infty$. Then ϑ is locally homogeneous at $g(x)$:

$$\forall\delta>0, \quad z\in H_\delta(0)\Rightarrow \vartheta(g(x)+z)=p_g(x)\,C_\vartheta z^{\alpha_\vartheta}+O(z^{\alpha_\vartheta+1})$$

and the multiplier p_g is given by (5.13).

Proof. Again it will be enough to verify that the remainder is well behaved under U. Let $\lambda=Dg(x)$. Since U is differentiable at $x\neq 0$, we have $U(x+z)=U(x)+\lambda z'$ for $z\in H_\delta(0)$ and some $z'\in H_{\delta'}(0)$. In addition, we have $z=z'+O(z^2)$. Therefore we can write

$$\vartheta(U(x)+\lambda z')=\vartheta(U(x+z))=f_U(x+z)\,\vartheta(x+z)$$

Using the hypothesis on the behaviour of ϑ at x, we can write

$$\vartheta(U(x)+\lambda z')=f_U(x+z)[C_\vartheta z^{\alpha_\vartheta}+O(z^{\alpha_\vartheta+1})]$$

Now f_U is differentiable at x and therefore

$$\begin{aligned} \vartheta(U(x)+\lambda z') &= [f_U(x)+O(z)][C_\vartheta z^{\alpha_\vartheta}+O(z^{\alpha_\vartheta+1})] \\ &= f_U(x)\,C_\vartheta z^{\alpha_\vartheta}+O(z^{\alpha_\vartheta+1}) \\ &= f_U(x)\,C_\vartheta z'^{\alpha_\vartheta}+O(z'^{\alpha_\vartheta+1}) \end{aligned}$$

If we now replace $\lambda z'$ by z, we find the stated result. $\blacksquare$

Remark. Let us consider instead of (5.1) and (5.12) any other analogous quantity, which can be obtained by replacing at least one of the inf by sup. Then we find that Lemmas 1 and 2 will hold for these quantities, too.

5.3. Some Scalings of ϑ

Until now we have shown how some local quantities transform under the theta group G_ϑ. We now come to calculate some of these quantities at some points. We will consider the following sets: the orbit of 0 under G_ϑ: $S = \{x \in \Re \mid \exists g \in G_\vartheta,\ x = g(0)\}$; the orbit of 1 under G_ϑ: $P = \{x \in \Re \mid \exists g \in G_\vartheta,\ x = g(1)\}$; and the nondegenerate fixed points of G_ϑ: $F = \{x \in \Re \mid \exists g \in G_\vartheta,\ g(x) = x,\ Dg(x) \neq 1\}$. These sets will be treated separately.

5.3.1. The Orbit of 0 under G_ϑ. First note that from (4.8b) one can obtain the behavior of ϑ at $i\infty$:

$$\vartheta(z) = 1 + O(e^{-\pi \operatorname{Im} z}) \qquad (\operatorname{Im} z \to \infty)$$

and so, because U will transform a path going to $i\infty$ into one going to 0, we find by (5.5) the behavior of ϑ near $x = 0$:

$$\begin{aligned} \forall \delta > 0\ z \in H_\delta(0) \Rightarrow \vartheta(z) &= \vartheta(U(-1/z)) = (-iz)^{-1/2}\,\vartheta(-1/z) \\ &= (-iz)^{-1/2} + O(e^{-\pi \operatorname{Im} 1/z}) \\ &= (i/z)^{1/2} + O(z^n) \qquad \text{for all} \quad n > 0 \end{aligned}$$

Therefore we can apply Lemma 3 to find the behavior of ϑ near any point of S. As before, (5.9), we can find a relation between Dg and p_g, (5.13). Since $\alpha = -1/2$, we have $p_g(x) = \zeta\,|Dg(x)|^{1/4}$, with some $\zeta^8 = 1$. Writing now $g(0) = b/d$ and $Dg(0) = 1/d^2$, we find

$$\begin{aligned} &b/d \in S, \text{ and } d \text{ relatively prime} \Rightarrow \\ &\qquad \vartheta(b/d + z) = \zeta\,|d|^{-1/2}\,(i/z)^{1/2} + O(z^{1/2}), \qquad \zeta^8 = 1 \end{aligned} \tag{5.14}$$

Again z should approach 0 in a cone $H_\delta(0)$. So we see that ϑ is approximately homogeneous around every point of S. The explicit dependence of ζ on b and d is rather complicated[6] and so we only give the transformation behavior of $\phi = \arg \zeta$ under T^2 and U:

$$\phi(T^2 x) = \phi(x), \qquad \phi(Ux) = \phi(x) - \operatorname{sign}(x)\,\pi/4 \tag{5.15}$$

Together with $\phi(0) = 0$, this determines ϕ along S.

This gives an explanation for the remarkable shape of Fig. 2d: let us again consider the renormalization of ϑ near the real axis, h_λ, that we have already met in the previous section, (4.9). Since $1/2 \in S$, and, on the other hand, ϑ does not diverge faster than $\operatorname{Im} z^{-1/2}$, (5.2), we find that

(i) The sequence $h_\lambda(x)$ $(\lambda \to 0)$ is bounded for any real x.

(ii) $b/d \in S$, b and d relatively prime $\Rightarrow h_\lambda(b/d) \to h(b/d) = h(1/2)\,|d|^{1/2}$

So, fixing b and letting d grow to infinity, we find that for each b there is a sequence of peaks scaling down at $x = 0$ with an exponent 1/2 in the following sense: let t be the distance of the position of the peak from $x = 0$, and let h be its height. Then $h \approx t^{1/2}$. Different values of b give rise to different hierarchies of peaks. But $x = 0$ is no better than any other point of S, since any $g \in G_\vartheta$, $g \neq U$, will transform a hierarchy of peaks scaling down at zero into one scaling down at $g(0)$, as can be seen with the help of Lemma 3. So there will be at any point on S an infinity of hierarchies of peaks, each scaling down with an exponent 1/2. This explains the whole self-similarity of Fig. 2g.

5.3.2. The Orbit of 1 under G_ϑ. We now come to the behavior of ϑ near the points that form the set P. Here we have the following result, which we state without proof[8]:

$$\forall \delta > 0\ \forall n : z \in H_\delta(0) \Rightarrow \vartheta(1+z) = O(z^n) \tag{5.16}$$

This shows that ϑ tends to 0 at any point of the orbit of 1 under G_ϑ.

Since the orbit of 0 under the whole modular group G is $\mathbb{Q}$, the coset decomposition (5.6) shows that $\mathbb{Q}$ is the union of S (the orbit of 0) and P (the orbit of 1). Since the scaling exponent is invariant under G_ϑ, we find that $\mathbb{Q}$ is the disjoint union of S and P, and so we know the scaling behavior of ϑ at any rational point.

5.3.3. The Nondegenerate Fixed Points of G_ϑ. We now come to the set $F = \{x \in \Re \mid \exists g \in G_\vartheta,\ g(x) = x,\ Dg(x) \neq 1\}$. Obviously for any x in F we have $Dg(x) \neq 0$. Without loss of generality we may assume that $Dg(x) < 1$ (if not, we consider g^{-1}). Then there is an open attracting domain $B \subset \mathbb{C}$, $x \in B$, such that the sequence $\{g^n z\}$, $z \in B$, converges to x:

$$z \in B \Rightarrow z_n = g^n(z) \to x \qquad (n \to \infty) \tag{5.17}$$

On the other hand, the sequence of theta values will be

$$\vartheta_n = \vartheta(z_n) = f_g(z_{n-1})\, f_g(z_{n-2}) \cdots f_g(x)\, \vartheta(z) \tag{5.18}$$

Both sequences (5.17) and (5.18) become geometric progressions as n

grows to infinity, with growth factors $Dg(x)$ and $f_g(x)$, respectively, and so we would expect that the local scaling exponent of ϑ at these points is

$$\lim_{n \to \infty} \log \vartheta(z_n)/\log(z_n - x) = \log f_g(x)/\log Dg(x) = -1/4 + i\gamma \quad (5.19)$$

with $\gamma \bmod 2\pi = \operatorname{Im} \log f_g(x)/\log Dg(x)$.

In fact, this heuristic argument is essentially true.

Theorem. For all $x \in F$, $g(x) = x$, we have

$$\lim_{z \to x} \log \vartheta(z)/\log(z - x) = -1/4 + i\gamma$$

where $\gamma = \operatorname{Im} r_g(x)/\log Dg(x)$, and r_g is given by (5.10). The point x should be approached in a cone $H_\delta(x)$.

Proof. We first need the following technical lemma, which shows that in a neighborhood of a fixed point, a holomorphic mapping is essentially given by its linear part.

Lemma. Let $g(x) = \lambda e^{i\phi} x + O(x^2)$ be holomorphic at $x = 0$, and $0 < \lambda < 1$. Then $\forall \varepsilon > 0$, $0 < \lambda - \varepsilon$, $\lambda + \varepsilon < 1$ there exists a neighborhood B of 0 such that:

(i) $\forall z \in B, \forall n$: $(\lambda - \varepsilon)^n |z| \leqslant |g^n(z)| \leqslant (\lambda + \varepsilon)^n |z|$

(ii) $\forall z \in B, z \neq 0, \forall n$: $\arg z + n\phi - \varepsilon \leqslant \arg g^n(z) \leqslant \arg z + n\phi + \varepsilon$

Proof. Because g is differentiable, we have $\lim_{x \to 0} |g(x)|/|x| = \lambda$, and so, given $\varepsilon > 0$, $0 < \lambda - \varepsilon$, $\lambda + \varepsilon < 1$, there is a disk A with center in 0 such that

$$x \in A \Rightarrow (\lambda - \varepsilon)\,|x| \leqslant |g(x)| \leqslant (\lambda + \varepsilon)\,|x|$$

Then note that for all n, the image $g^n A$ of A under g^n is in A, and so we can conclude the proof of (i) by an induction argument (we only write the right-hand side):

$$|g^{n+1}(z)| = |g(g^n(z))| \leqslant (\lambda + \varepsilon)\,|g^n(z)| \leqslant (\lambda + \varepsilon)^{n+1}\,|z|$$

To prove (ii), we note that $\arg g(z) = \phi + \arg z + O(z)$ and therefore, given $\varepsilon > 0$, $0 < \lambda - \varepsilon$, $\lambda + \varepsilon < 1$, there is another disk $B \subset A$ with center at 0, whose radius is smaller than ε, and a constant $c > 0$ such that:

(j) Part (i) of the lemma holds for B.

(jj) $z \in B \Rightarrow \phi + \arg z - c|z| \leqslant \arg z \leqslant \phi + \arg z + c|z|$.

From part (i) of the lemma it follows that $\max |g^p B| \leqslant \varepsilon(\lambda+\varepsilon)^p$, and so again an induction argument will help us (we only write the right-hand side):

$$\begin{aligned}
\arg g^{n+1}(z) = \arg g(g^n(z)) &\leqslant \arg g^n(z) + \phi + c \max |g^n B| \\
&\leqslant \arg z + (n+1)\,\phi + \varepsilon c \sum_{p=1,n} (\lambda+\varepsilon)^p \\
&\leqslant \arg z + (n+1)\,\phi + \varepsilon c \sum_{p=1,\infty} (\lambda+\varepsilon)^p \\
&\leqslant \arg z + (n+1)\,\phi + \varepsilon c_{te}
\end{aligned}$$

Since ε was arbitrary, the lemma is proved. ∎

Now we come to the proof of the theorem. Let $x \in F$ be the real fixed point of $g \in G_\vartheta$. Then g has a positive real derivative, which, without loss of generality, we can assume to be smaller than 1. Therefore the lemma holds with $\lambda = Dg(x)$ and $\phi = 0$ provided we translate all functions in question: so we pose $\Psi(z) = \vartheta(x+z)$, $h(z) = g(x+z) - x$, and $b_h(z) = r_g(x+z)$. Then we have

$$\log \Psi(h(z)) = b_h(z) + \log \Psi(z) \tag{5.20}$$

and we are interested in the following limit:

$$\lim_{z \to 0} \log \Psi(z)/\log z \tag{5.21}$$

Let K be a compact in H such that $h^n K$ converges to zero as n grows to infinity. Then we will prove that the following limit exists uniformly in K:

$$\lim_{n\to\infty} X_n(K) = \lim_{n\to\infty} \log \Psi(h^n K)/\log h^n K = -1/4 + i\gamma \tag{5.22}$$

with γ as stated in the theorem.

Let $\kappa = \lim_{z\to 0} b_h(z) = r_g(x)$. Given any $\varepsilon > 0$ (small enough), there is a disk D with center at $x = 0$ such that (we denote by E the unit disk around $x = 0$):

$$\begin{aligned}
&\text{(i)}\quad \text{the lemma holds for } \varepsilon \text{ and } D \\
&\text{(ii)}\quad b_h(D \cap H) \subset \kappa + \varepsilon E
\end{aligned} \tag{5.23}$$

Here and in the following, the operations on the sets should be understood in the natural way. Then, since the compact K lies in the attracting domain of $x = 0$, there is an integer m such that

$$h^m(K) \subset D \tag{5.24}$$

Consequently, we may suppose that $K \subset D$ [because $X_n(h^m(K)) = X_{n+m}$]. Using the covariance of Ψ, (5.20), we write

$$X_n(K) = \left[\log \Psi(K) + \sum_{p=1,n} b_h(h^{p-1}K)\right] \Big/ \log h^n(K) \tag{5.25}$$

We now apply the lemma to estimate $\log h_n(K)$ [$\lambda = Dh(0)$]:

$$\forall n > 0\colon \log h^n(K) \subset n \log(\lambda + \varepsilon E) + \varepsilon E + \log K \tag{5.26}$$

Then there is an n_0 such that for all $n > n_0$ we have:

$$\begin{aligned} &\text{(i)} \quad \log \Psi(K)/\log h^n(K) \subset \varepsilon E \\ &\text{(ii)} \quad \log K/n \in \varepsilon E \end{aligned} \tag{5.27}$$

The first is clear, since K is a compact in H, and ϑ—and therefore Ψ—is never 0 in H.[7] The second is clear, since K is bounded away from zero.

All this together, (5.23)–(5.27), yields

$$n > n_0 \Rightarrow X_n(K) \subset 2\varepsilon E + (\kappa + \varepsilon E)/[\log(\lambda + \varepsilon E) + 2\varepsilon E]$$

Now ε was arbitrary, which shows that the limit (5.22) exists and is equal to $\kappa/\log \lambda$. From relation (5.9) it follows easily that $\operatorname{Re} \kappa = -1/4$, and so the assertion (5.22).

If we now show that for all $\delta > 0$ there is a compact K, $K \subset H$, and a disk D of radius small enough, with center at 0, which satisfies

$$H_\delta(0) \cap D \subset \bigcup_{p=1,\infty} h^p(K) \tag{5.28}$$

we would have finished the proof, since for any n, any path going to (0.0) in H_δ will finally stay in $\bigcup_{p=n,\infty} h^p(K)$.

We now want to construct this compact K. First note that there is a disk D such that the lemma holds with some $\varepsilon < \min\{Dh(0),\ 1-\lambda,\ \delta/2\}$, and that h restricted to D is injective. Then we have $hD \subset D$ and so by iteration $h^{n+1}D \subset h^n D$. Then let K_1 be the compact closure of $D \backslash hD$. A moment's reflection shows that

$$\bar{D} = \overline{\bigcup_{p=1,\infty} h^p(K_1)}$$

Now let $K = K_1 \cap H_\varepsilon(0)$. Part (ii) of the lemma shows that $\delta < \arg h^n K < \pi - \delta$, which proves (5.28), and therefore the theorem.

5.4. Back to W_β

We now interpret our results in terms of the fractal family W_β. We have found three different scalings to which there are associated the three sets S, R, and F. Throughout this section we supose, since we could not prove it, that W_β satisfies locally condition (ES) or (PS) as defined in Section 3. We now present results about W_β from the analysis of ϑ for the three different classes of points.

5.4.1. The Orbit of 0 under G_ϑ. At the points that form the set S, the functions W_β might show exact scaling behavior. From the relation between ϑ and the wavelet transform T_β of W_β, (4.7), it follows that the local scaling exponent $\alpha_\beta(x)$ of W_β at these points is

$$\alpha_\beta(x) = (\beta - 1)/2 \tag{5.29}$$

whereas the periodic scaling exponent γ_β is zero. The local constants c_+ and c_- as given by the definition (ES) can now be calculated from (3.7) and (4.4). For b/d in S, and b, d relatively prime, and $\beta \neq 3, 5,\ldots$, we have

$$\begin{aligned} c_+(b/d) &= K|d|^{-1/2} \sin[\pi(\beta+1)/4 - \phi]/\sin[\pi(\beta+1)/2] \\ c_-(b/d) &= K|d|^{-1/2} \sin[\pi(\beta+1)/4 + \phi]/\sin[\pi(\beta+1)/2] \end{aligned} \tag{5.30}$$

where the phase ϕ is determined by (5.15). The constant is given by

$$K = (1/4)\, \pi^{\beta/2+1}/\Gamma([\beta+1]/2) \tag{5.31}$$

So we see that there are eight kinds of local symmetry of W_β at the points that form the set S. This is obviously due to the eighth root of the unity in (5.14). At $x = 0$ we rediscover that W_β is locally even. For $1 < \beta < 3$ we find that W_β has a cusp at $x = 0$ pointing to infinity; for $\beta = 2$ the function is differentiable at the left at $x = 1/2$ (compare Fig. 1).

5.4.2. The Orbit of 1 under G_ϑ. At these points the local exponents are no longer integrable with respect to the wavelet $g_{\beta/2}$. So we only are able to give a lower bound for $\alpha_\beta(x)$ for $x \in P$:

$$\alpha_\beta(x) \geqslant \beta/2 \tag{5.32}$$

Notice that for $\beta = 2$ this could imply that W_β is differentiable at these points.

5.4.3. The Nondegenerate Fixed Points. At these points W_β shows oscillatory critical behavior. The local scaling exponent is then the same for all $x \in F$:

$$\alpha_\beta(x) = (2\beta - 1)/4 \tag{5.33}$$

The periodic scaling exponent γ_β will depend on x, but will be independent of β. It is given by the theorem that we just have proved.

6. DISCUSSION

We have shown that the wavelet transformation may be useful for analyzing the scaling behavior of fractals. For two special classes of fractals, we were able to give rigorous results. However, we were not able to characterize uniquely scaling properties with the help of this transformation. We hope to be able to report on this in a forthcoming article. In the case of a special family of fractals W_β, we calculated explicitly the wavelet transform, which in this case was a Jacobi theta function. We analyzed some scalings of this theta function, which we finally used to obtain some indications about the scaling behavior of W_β. It might be interesting to analyze other fractals associated to modular functions, on which we hope to report soon.

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